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Poincaré–Birkhoff–Witt theorems and generalized Casimir invariants for some infinite-dimensional Lie groups: II

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Abstract

In a previous paper we gave a generalization of the notion of Casimir invariant differential operators for the infinite-dimensional Lie groups $\mathrm{GL}^{\infty}(\mathbb{C})$ (or equivalently, for its Lie algebra $\mathfrak{gl}^{\infty}(\mathbb{C})$). In this paper we give a generalization of the Casimir invariant differential operators for a class of infinite-dimensional Lie groups (or equivalently, for their Lie algebras) which contains the infinite-dimensional complex classical groups. These infinite-dimensional Lie groups, and their Lie algebras, are inductive limits of finite-dimensional Lie groups, and their Lie algebras, with some additional properties. These groups or their Lie algebras act via the generalized adjoint representations on projective limits of certain chains of vector spaces of universal enveloping algebras. Then the generalized Casimir operators are the invariants of the generalized adjoint representations. In order to be able to explicitly compute the Casimir operators one needs a basis for the universal enveloping algebra of a Lie algebra. The Poincaré–Birkhoff–Witt (PBW) theorem gives an explicit construction of such a basis. Thus in the first part of this paper we give a generalization of the PBW theorem for inductive limits of Lie algebras. In the last part of this paper a generalization of the very important theorem in representation theory, namely the Chevalley–Racah theorem, is also discussed.

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1. Introduction

In [13] we introduced a notion of Casimir invariant differential operators of the infinite-dimensional Lie group $\mathrm{GL}^{\infty}(\mathbb{C})$ (or equivalently, of its Lie algebra $\mathfrak{gl}^{\infty}(\mathbb{C})$). This paper is a continuation of [13]. Recall that in physics, if G is a symmetry group of some physical system, then the spectra of the Casimir invariant differential operators determine the observable quantum numbers of the physical system. The important Chevalley–Racah theorem states that: '*for every semi-simple Lie algebra* $\mathfrak{g}$ *of rank r there exists an algebraically independent set of r generators for the algebra of Casimir invariants, whose spectra* (*eigenvalues*) *characterize the finite-dimensional irreducible representations of* $\mathfrak{g}$' (see [1, chapter 9]). In the context of mathematics, the classical Casimir invariant differential operators can be described more precisely as follows. Let G be a connected Lie group of dimension n with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be defined as the set of all left- (or right-) invariant analytic vector fields on G. Let $\mathfrak{U}$ denote the universal enveloping algebra of $\mathfrak{g}$. Then $\mathfrak{U}$ can be identified with the algebra of all left- (or right-) invariant analytic differential operators on G. The adjoint representation of G (resp., $\mathfrak{g}$) on $\mathfrak{g}$ extends uniquely to an action of G (resp., $\mathfrak{g}$) on $\mathfrak{U}$, which we shall denote by $\mathrm{Ad}(g)$, $g \in G$ (resp., $\mathrm{ad}(X)$, $X \in \mathfrak{g}$). Then an element $u \in \mathfrak{U}$ is called a *Casimir invariant differential operator* if $\mathrm{Ad}(g)u = u$ for all $g \in G$. Since G is connected, an easy argument shows that this is equivalent to $\mathrm{ad}(X)u = 0$ for all $X \in \mathfrak{g}$. But $\mathrm{ad}(X)u = Xu - uX$, so that the latter implies, in turn, that u is a Casimir invariant if and only if u belongs to the centre of $\mathfrak{U}$. Casimir introduced the quadratic Casimir invariant, and Chevalley, Gelfand and Harish-Chandra gave the general definition above.

Now suppose that G_0 is a compact, connected, semi-simple Lie group. Let T be a maximal torus of G_0. Let G be the complexification of G_0 and B a Borel subgroup of G; then $G = BG_0$ and $B \setminus G = T \setminus G_0$. If ξ is a character of T, then ξ extends uniquely to a holomorphic homomorphism of B into $\mathbb{C}^*$, which we shall denote by the same symbol ξ. Let $\mathcal{H}ol(G, \xi)$ be the space of all holomorphic functions f on G which also satisfy $f(bg) = \xi(b) f(g)$ for all $(b, g) \in B \times G$. It is well known that $\mathcal{H}ol(G, \xi)$ is finite dimensional. Let $\pi(\cdot, \xi)$ be the representation of G on $\mathcal{H}ol(G, \xi)$ which is given by $(\pi(g, \xi) f)(x) = f(x)$ for all $x, g \in G$. Then we have the Borel–Weil theorem:

(1) *The space* $\mathcal{H}ol(G, \xi)$ *is non-zero if and only if* ξ *is a dominant weight.*
(2) *The representation* $\pi(\cdot, \xi)$ *is irreducible if* ξ *is dominant. In this case, the restriction of* $\pi(\cdot, \xi)$ *to* G_0 *remains irreducible, and its highest weight is* ξ.

Note that a weight corresponds to an r-tuple of integers $(m) = (m_1, \ldots, m_r)$, and a weight is *dominant* if and only if the integers satisfy a certain condition; for example, for $G_0 = \mathrm{U}(r)$, (m) is dominant if and only if $m_1 \geqslant \cdots \geqslant m_r$. Thus by Weyl's 'unitarian trick' (see [16, section 4.11]) the Borel–Weil theorem gives a concrete realization of all irreducible unitary representations of a compact, connected, semi-simple Lie group.

Let π be an irreducible unitary representation of G_0 on a finite-dimensional complex vector space V. By the Borel–Weil theorem and the 'unitarian trick' π extends uniquely to a complex analytic irreducible representation of G, which we shall denote by the same symbol π. Then since G is connected, the differential $\mathrm{d}\pi$ is an irreducible representation of $\mathfrak{g}$ on V. The representation $\mathrm{d}\pi$ extends uniquely to an algebra homomorphism of $\mathfrak{U}(\mathfrak{g})$ into the associative algebra $\mathrm{End}(V)$ of all endomorphisms of V. Clearly the image of a Casimir operator under this homomorphism, which we shall also call a Casimir operator, commutes with $\mathrm{d}\pi(X)$, for all $X \in \mathfrak{g}$. But since G is connected, an operator commutes with all $\mathrm{d}\pi(X)$ if and only if it commutes with all $\pi(g)$, $g \in G$. Let C be a Casimir operator; then by Schur's lemma $C = \lambda I$, where I is the identity operator on V and $\lambda \in \mathbb{C}$ represents the spectrum of C. The Chevalley–Racah theorem states that λ is a function of the m_i and that there exist r Casimir operators $C_1, \ldots, C_r$ such that the integers m_i can be expressed as functions of the spectra λ_i.

In this paper we shall give a generalization of the Casimir invariant differential operators for a class of infinite-dimensional Lie group which contains the infinite-dimensional complex classical groups. These infinite-dimensional Lie groups, and their Lie algebras, are inductive limits of finite-dimensional Lie groups, and of their Lie algebras, with some additional properties (see section 3 for these properties). As discussed above, in the case of a finite-dimensional Lie group the Casimir operators form the centre of the universal enveloping algebra of its Lie algebra. And in order to be able to explicitly compute the Casimir operators one needs a basis for the universal enveloping algebra. The so-called Poincaré–Birkhoff–Witt (PBW for short) theorem gives an explicit construction of such a basis (see also [14, 7] for two historical accounts of this theorem). Thus in section 2 we give a generalization of the PBW theorem for inductive limits of Lie algebras. But there is a well-known theorem which states that '*the centre of the universal enveloping algebra of any infinite-dimensional simple Lie algebra over an arbitrary field* $\mathbb{K}$ *is* $\mathbb{K}I$ (*multiples of the identity operator*)'. Thus if one wants to give a generalization of operators that have all the properties of Casimir operators such as a Chevalley–Racah-type theorem for irreducible 'tame' representations of infinite-dimensional classical groups, etc, one must relax the condition that the Casimir operators must belong to the centre of the universal enveloping algebra. It turns out that this condition is not essential and the most natural context to generalize Casimir operators is that of *projective* (*or inverse*) *limits of universal enveloping algebras* (*considered as vector spaces and not as algebras*) *which are carrier spaces of representations. Note that these projective limits contain the inductive limits.* With this objective in mind, we developed in [15] a general theory of invariants of inductive limits of Lie groups (or equivalently, of their Lie algebras) acting on projective limits of vector spaces, modules, rings, or algebras. In fact, the proof of the main result of this paper (theorem 3.3) hinges on the fundamental theorem of the invariant theory of [15]. Surprisingly, as a consequence of theorem 3.3 we can give an ingenious proof (corollary 3.10) of the above-mentioned fact that the centre of the universal enveloping algebra of any infinite-dimensional simple complex classical group is trivial.

In [13, 9] we essentially generalized the Chevalley–Racah theorem for the group $\mathrm{GL}^{\infty}(\mathbb{C})$ (or equivalently, for its Lie algebra $\mathfrak{gl}^{\infty}(\mathbb{C})$). In this paper, using the results on classical Casimir invariants (cf, e.g., [1, 4, 16]) and the theorems in section 3 we can easily generalize this theorem to the case of irreducible tame representations of any infinite-dimensional simple complex classical groups (or equivalently, of their Lie algebras). This theorem was used in [9] to decompose tensor products of tame representations of $\mathrm{U}(\infty)$. The other cases will be considered in forthcoming papers. Other generalizations of Casimir invariants for infinite-dimensional classical groups may be found in [12, 6].

2. Poincaré–Birkhoff–Witt theorem for inductive limits of Lie algebras

Let $\mathbb{I}$ be a directed set, let $(\mathfrak{g}_\alpha)_{\alpha\in\mathbb{I}}$ be a family of Lie algebras over a field $\mathbf{k}$ of characteristic 0. Suppose that for every pair (α, β) of indices in $\mathbb{I}$ such that $\alpha \leqslant \beta$, there exists a *Lie algebra monomorphism* $f_{\beta\alpha}$ which satisfies the following condition:

$$\text{the relations } \alpha \leqslant \beta \leqslant \gamma \text{ imply } f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}, \qquad \forall \alpha, \beta, \gamma \in \mathbb{I}. \tag{2.1}$$

Let $\mathcal{F}$ denote the *sum* (or *free union*) of the family of sets $(\mathfrak{g}_\alpha)_{\alpha\in\mathbb{I}}$. The image of an $X_\alpha \in \mathfrak{g}_\alpha$ under any *connecting morphism* $f_{\beta\alpha}$ is termed a *successor of* X_α. Define the relation R on $\mathcal{F}$ by calling two elements $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$ *equivalent* if they have a common successor. It is straightforward to verify that R is an *equivalence relation* in $\mathcal{F}$. We denote the quotient set $\mathcal{F}/R$ by $\mathfrak{g}^\infty$ (or $\varinjlim \mathfrak{g}_\alpha$), and we call $\mathfrak{g}^\infty$ the *inductive limit* of the family $(\mathfrak{g}_\alpha)_{\alpha\in\mathbb{I}}$ relative to the family of connecting morphisms $f_{\beta\alpha}$. Moreover, the pair $(\mathfrak{g}_\alpha, f_{\beta\alpha})$ is called an inductive system of sets, relative to the index set $\mathbb{I}$.

Let f_α denote the restriction to $\mathfrak{g}_\alpha$ of the canonical map f of $\mathcal{F}$ onto $\mathfrak{g}^\infty$. Then f_α is called the *canonical map* of $\mathfrak{g}_\alpha$ onto $\mathfrak{g}^\infty$. It is straightforward to verify that

$$\text{for } \alpha \leqslant \beta \quad \text{we have} \quad f_\beta \circ f_{\beta\alpha} = f_\alpha. \tag{2.2}$$

We define a structure of Lie algebra on $\mathfrak{g}^\infty$ as follows:

For $X, Y \in \mathfrak{g}^\infty$ and $\lambda \in \mathbf{k}$, $\exists \alpha \in \mathbb{I}$ such that

$X = f_\alpha(X_\alpha), \qquad Y = f_\alpha(Y_\alpha), \quad X_\alpha, \qquad Y_\alpha \in \mathfrak{g}_\alpha.$

Define

$$\begin{cases} X + Y := f_\alpha(X_\alpha + Y_\alpha), \\ \lambda X := f_\alpha(\lambda X_\alpha), \\ [X, Y] := f_\alpha([X_\alpha, Y_\alpha]). \end{cases} \tag{2.3}$$

The fact that the $f_{\beta\alpha}$ are Lie algebra homomorphisms and relation (2.2) imply that these operations are well defined and the canonical maps f_α are Lie algebra homomorphisms. Let us show, for example, that Jacobi's identity holds in $\mathfrak{g}^\infty$.

First let us define 0, the neutral element for addition in $\mathfrak{g}^\infty$. Since $f_{\beta\alpha}(0_\alpha) = 0_\beta$ whenever $\alpha \leqslant \beta$, where 0_α (resp. 0_β) is the unique neutral element for addition in $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_\beta$), it follows from relation (2.2) that the unique element $0 \in \mathfrak{g}^\infty$ such that $0 = f_\alpha(0_\alpha)$ for all $\alpha \in \mathbb{I}$ is the neutral element for addition in $\mathfrak{g}^\infty$.

Now let X, Y, Z be three elements of $\mathfrak{g}^\infty$; then there exist $\alpha \in \mathbb{I}$ and three elements $X_\alpha, Y_\alpha, Z_\alpha$ in $\mathfrak{g}_\alpha$ such that $X = f_\alpha(X_\alpha)$, $Y = f_\alpha(Y_\alpha)$, $Z = f_\alpha(Z_\alpha)$. The fact that f_α is a Lie algebra homomorphism implies that

$$\begin{aligned}[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= f_\alpha([X_\alpha, [Y_\alpha, Z_\alpha]] + [Y_\alpha, [Z_\alpha, X_\alpha]] + [Z_\alpha, [X_\alpha, Y_\alpha]]) \\ &= f_\alpha(0_\alpha) = 0.\end{aligned}$$

By assumption, the homomorphism $f_{\beta\alpha}$ is injective for $\alpha \leqslant \beta$. It follows that each Lie algebra $\mathfrak{g}_\alpha$ can be identified with a Lie subalgebra of $\mathfrak{g}_\beta$ for $\alpha \leqslant \beta$. Therefore, 0_α is identified with 0_β for $\alpha \leqslant \beta$. Hence, every homomorphism f_α is injective, and thus every $\mathfrak{g}_\alpha$ can be identified with a subalgebra of $\mathfrak{g}^\infty$.

From the discussion above we can infer that there exists a basis $\mathcal{B} = \{X_i : i \in J\}$ of $\mathfrak{g}^\infty$, where J is a linearly ordered set, such that for every α there exists a subset J_α of J with $J_\alpha \subset J_\beta$ whenever $\alpha \leqslant \beta$ and that $\mathcal{B}_\alpha = \{X_i : i \in J_\alpha\}$ is a basis of $\mathfrak{g}_\alpha$.

Let us recall the functorial definition of *the universal enveloping algebra of a Lie algebra*.

A pair $(\mathfrak{U}, \pi)$, where $\mathfrak{U}$ is a unital associative algebra over $\mathbf{k}$ and π is a *linear mapping* of the Lie algebra $\mathfrak{g}$ into $\mathfrak{U}$, is called a universal enveloping algebra of $\mathfrak{g}$ if the following conditions are satisfied:

(i) $\pi(\mathfrak{g})$ generates $\mathfrak{U}$,
(ii) $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$, $\forall X, Y \in \mathfrak{g}$,
(iii) if $\mathfrak{A}$ is a unital associative algebra over $\mathbf{k}$ and ξ a linear map of $\mathfrak{g}$ into $\mathfrak{A}$ such that

$$\xi([X, Y]) = \xi(X)\xi(Y) - \xi(Y)\xi(X), \qquad \forall X, Y \in \mathfrak{g},$$

then there exists a unique homomorphism ξ' of $\mathfrak{U}$ into $\mathfrak{A}$ such that $\xi(X) = \xi'(\pi(X))$, $\forall X \in \mathfrak{g}$.

It can be shown that there is a unique (up to isomorphism) universal enveloping of the Lie algebra $\mathfrak{g}$, which we shall refer to as the universal enveloping algebra of $\mathfrak{g}$ (see, e.g., [16, chapter 3]).

Theorem 2.1 (The Poincaré–Birkhoff–Witt theorem for inductive limits of Lie algebras)**.** *We preserve the notation and assumptions introduced above. If $(\mathfrak{U}_\alpha, \pi_\alpha)$ denotes the universal* *enveloping algebra of* $\mathfrak{g}_\alpha$ $(\forall\alpha \in \mathbb{I})$, *then there exist connecting morphisms* $\varphi_{\beta\alpha} : \mathfrak{U}_\alpha \to \mathfrak{U}_\beta$ $(\alpha \leqslant \beta)$ *such that* $(\mathfrak{U}_\alpha, \varphi_{\beta\alpha})$ *is an inductive system relative to the index set* $\mathbb{I}$. *Let* $\mathfrak{U}^\infty = \varinjlim \mathfrak{U}_\alpha$ *and* $\varphi_\alpha : \mathfrak{U}_\alpha \to \mathfrak{U}^\infty$ *the canonical map. Then*

(i) $\mathfrak{U}^\infty$ *is an associative algebra spanned by 1 and the products* $\varphi_\alpha(\pi_\alpha(X_{i_1}) \cdots \pi_\alpha(X_{i_s}))$ $(i_1, \dots, i_s \in J, s \geqslant 1, \alpha \in \mathbb{I})$,
(ii) $\mathfrak{U}^\infty$ *is the universal enveloping algebra of* $\mathfrak{g}^\infty$.

Proof. (i) For $\alpha \leqslant \beta$ we have the following diagram:

$$\begin{array}{ccccc} \mathfrak{g}_\alpha & \xrightarrow{f_{\beta\alpha}} & \mathfrak{g}_\beta & \xrightarrow{\pi_\beta} & \mathfrak{U}_\beta. \\ {\scriptstyle\pi_\alpha}\downarrow & & & & \\ \mathfrak{U}_\alpha & & & & \end{array}$$

Since both $f_{\beta\alpha}$ and π_β are monomorphisms, $\pi_\beta \circ f_{\beta\alpha}$ is a monomorphism. We also have

$$\begin{aligned} \pi_\beta(f_{\beta\alpha}[X_\alpha, Y_\alpha]) &= \pi_\beta([f_{\beta\alpha}(X_\alpha), f_{\beta\alpha}(Y_\alpha)]) \\ &= \pi_\beta(f_{\beta\alpha}(X_\alpha))\pi_\beta(f_{\beta\alpha}(Y_\alpha)) - \pi_\beta(f_{\beta\alpha}(Y_\alpha))\pi_\beta(f_{\beta\alpha}(X_\alpha)) \end{aligned}$$

for all $X_\alpha, Y_\alpha \in \mathfrak{g}_\alpha$. By the functorial property of the universal enveloping algebra of $\mathfrak{g}_\alpha$ there exists one and only one homomorphism (necessarily injective) $\varphi_{\beta\alpha} : \mathfrak{U}_\alpha \to \mathfrak{U}_\beta$ which is an extension of $f_{\beta\alpha}$ and such that $\varphi_{\beta\alpha}(1) = 1$. Then it is straightforward to verify that $(\mathfrak{U}_\alpha, \varphi_{\beta\alpha})$ is an inductive system relative to the index set $\mathbb{I}$, that $\mathfrak{U}^\infty = \varinjlim \mathfrak{U}_\alpha$ is a unital associative algebra, and the canonical maps $\varphi_\alpha : \mathfrak{U}_\alpha \to \mathfrak{U}^\infty$ are algebra monomorphisms. Now since each $\mathfrak{U}_\alpha$ is spanned by 1 and $\pi_\alpha(X_{i_1}) \cdots \pi(X_{i_s})$ $(i_1, \dots, i_s \in J_\alpha, s \geqslant 1)$ and $J_\alpha \subset J_\beta$ whenever $\alpha \leqslant \beta$, it follows that $\mathfrak{U}^\infty$ is spanned by 1 and the products

$$\varphi_\alpha\big(\pi_\alpha\big(X_{i_1}\big) \cdots \pi_\alpha\big(X_{i_s}\big)\big) \qquad (i_1, \dots, i_s \in J, s \geqslant 1, \alpha \in \mathbb{I}).$$

(ii) For $\alpha \leqslant \beta$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g}_\alpha & \xrightarrow{\pi_\alpha} & \mathfrak{U}_\alpha \\ {\scriptstyle f_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle \varphi_{\beta\alpha}}. \\ \mathfrak{g}_\beta & \xrightarrow[\pi_\beta]{} & \mathfrak{U}_\beta \end{array}$$

So there exists a unique mapping $\pi : \mathfrak{g}^\infty \to \mathfrak{U}^\infty$ such that the diagram

$$\begin{array}{ccc} \mathfrak{g}_\alpha & \xrightarrow{\pi_\alpha} & \mathfrak{U}_\alpha \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle \varphi_\alpha} \\ \mathfrak{g}^\infty & \xrightarrow[\pi]{} & \mathfrak{U}^\infty \end{array}$$

commutes, for all $\alpha \in \mathbb{I}$ (see [2, p 19]). Then it follows immediately from part (i) that $\pi(\mathfrak{g}^\infty)$ generates $\mathfrak{U}^\infty$ and $\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$ for all $X, Y \in \mathfrak{g}^\infty$. Now if $\mathfrak{A}$ is a unital associative algebra over $\mathbf{k}$ and ξ is a linear map from $\mathfrak{g}^\infty$ into $\mathfrak{A}$ such that $\xi([X, Y]) = \xi(X)\xi(Y) - \xi(Y)\xi(X)$ for all $X, Y \in \mathfrak{g}^\infty$, then by the functorial property of $(\mathfrak{U}_\alpha, \pi_\alpha)$ there exists an algebra homomorphism θ_α such that the following diagram

$$\begin{array}{ccc} \mathfrak{g}_\alpha & \xrightarrow{\pi_\alpha} & \mathfrak{U}_\alpha \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle \theta_\alpha} \\ \mathfrak{g}^\infty & \xrightarrow[\xi]{} & \mathfrak{A} \end{array}$$

commutes for all $\alpha \in \mathbb{I}$. It follows that there exists a unique algebra homomorphism $\xi' : \mathfrak{U}^\infty \to \mathfrak{A}$ such that the diagram

$$\begin{array}{ccccc} \mathfrak{g}_\alpha & \xrightarrow{\pi_\alpha} & \mathfrak{U}_\alpha & \xrightarrow{\varphi_\alpha} & \mathfrak{U}^\infty \\ f_\alpha \downarrow & & \pi & & \downarrow \xi' \\ \mathfrak{g}^\infty & & \xrightarrow[\xi]{\hspace{4em}} & & \mathfrak{A} \end{array}$$

commutes for all $\alpha \in \mathbb{I}$. By the functorial property of $(\mathfrak{U}^\infty, \pi)$, $\mathfrak{U}^\infty$ is the universal enveloping algebra of $\mathfrak{g}^\infty$. □

Example 2.2. Consider the Bargmann–Segal–Fock space $\mathcal{F}(\mathbb{C}^{n\times N})$. Let L denote the action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{F}(\mathbb{C}^{n\times N})$ defined by

$$(L(g)f)(z) = f(g^{-1}z), \qquad g \in \mathrm{GL}_n(\mathbb{C}), \quad f \in \mathcal{F}(\mathbb{C}^{n\times N}), \quad z \in \mathbb{C}^{n\times N}. \tag{2.4}$$

Then a system of generators of the infinitesimal action of L is given by

$$L_{\alpha\beta} = \sum_{i=1,\dots,N} z_{\alpha i} \frac{\partial}{\partial z_{\beta i}}, \qquad 1 \leqslant \alpha, \quad \beta \leqslant n. \tag{2.5}$$

Let H_n denote the Heisenberg group of $2n + 1$ generators. Then a representation of the semidirect product of the Lie algebras of $\mathrm{GL}_n(\mathbb{C})$ and H_n on $\mathcal{F}(\mathbb{C}^{n\times N})$ can be given by the following generators (see [10] for details):

$$\begin{cases} L_{\alpha\beta}, \quad 1 \leqslant \alpha, \quad \beta \leqslant n, \quad z_{\gamma N}, \quad \dfrac{\partial}{\partial z_{\gamma N}}, \quad 1 \leqslant \gamma \leqslant n, \text{ where} \\ [L_{\alpha\beta}, L_{\lambda\mu}] = \delta_{\beta\lambda} L_{\alpha\mu} - \delta_{\alpha\mu} L_{\lambda\beta}, \quad 1 \leqslant \lambda, \mu \leqslant n, \\ [L_{\alpha\beta}, z_{\gamma N}] = \delta_{\beta\gamma} z_{\alpha N}, \quad \left[L_{\alpha\beta}, \dfrac{\partial}{\partial z_{\gamma N}}\right] = -\delta_{\alpha\gamma} \dfrac{\partial}{\partial z_{\beta N}}, \\ [z_{\alpha N}, z_{\beta N}] = 0, \quad \left[\dfrac{\partial}{\partial z_{\alpha N}}, \dfrac{\partial}{\partial z_{\beta N}}\right] = 0, \quad \left[\dfrac{\partial}{\partial z_{\alpha N}}, z_{\beta N}\right] = \delta_{\alpha\beta} I. \end{cases} \tag{2.6}$$

Let $\mathfrak{g}_n$ denote the Lie algebra generated by these operators. Let $\mathbb{I} = \mathbb{N}$ denote the index set of natural numbers. For $n, m \in \mathbb{I}$ with $n \leqslant m$ let $f_{mn} : \mathfrak{g}_n \to \mathfrak{g}_m$ denote the imbedding of $\mathfrak{g}_n$ in $\mathfrak{g}_m$. Then the pair $(\mathfrak{g}_n, f_{mn})$ is obviously an inductive system of Lie algebras relative to the index set $\mathbb{I}$. Let $\mathfrak{g}^\infty = \varinjlim \mathfrak{g}_n$; then $\mathfrak{g}^\infty$ is a Lie algebra and theorem 2.1 holds for $\mathfrak{g}^\infty$.

3. Generalized Casimir operators for a class of infinite-dimensional groups

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space with a fixed basis $\{e_1, \dots, e_k, \dots\}$. Let $\mathrm{GL}_k(\mathbb{C})$ denote the group of all invertible bounded linear operators on $\mathcal{H}$ which leave the vectors $e_n, n > k$, fixed. We define $\mathrm{GL}^\infty(\mathbb{C})$ as the inductive limit of the ascending chain of subgroups

$$\mathrm{GL}_1(\mathbb{C}) \subset \cdots \subset \mathrm{GL}_k(\mathbb{C}) \subset \cdots.$$

Let $\mathbb{I}$ be an infinite subset of the natural numbers $\mathbb{N}$. If for each $k \in \mathbb{I}$ we have a Lie subgroup G_k of $\mathrm{GL}_k(\mathbb{C})$ such that G_k is naturally imbedded in G_l, $l \geqslant k$, then we can define the *inductive limit* $G^\infty = \varinjlim G_k = \bigcup_{k\in\mathbb{I}} G_k$. Then G^∞ can be given a topological group structure in which the descending chain of subgroups of the type $\left\{\begin{bmatrix} 1_k & 0 \\ 0 & * \end{bmatrix}\right\}$ constitute a fundamental system of neighbourhoods of the identity 1_∞ (cf [11]).

Definition 3.1 (see [8])**.** *Suppose G is a connected Lie group and $H \subset G$ a closed subgroup. Let $\mathfrak{g} \supset \mathfrak{h}$ be their respective Lie algebras and $\mathfrak{m}$ a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the homogeneous space $H\backslash G$ is called reductive if the subspace $\mathfrak{m}$ can be chosen such that*

$$\mathrm{Ad}_G(h)\mathfrak{m} \subset \mathfrak{m} \qquad (h \in H), \tag{3.1}$$

where Ad_G denotes the adjoint representation of G on $\mathfrak{g}$.

Examples 3.2. (i) Let $G = \mathrm{GL}_l(\mathbb{C})$, $H = \mathrm{GL}_k(\mathbb{C}), k \leqslant l$. Then H is identified with the closed subgroup of G of the form

$$\left\{ h \equiv \begin{pmatrix} 1_{l-k} & 0 \\ 0 & h \end{pmatrix} \,\middle|\, h \in \mathrm{GL}_k(\mathbb{C}) \right\}.$$

Then $\mathfrak{g} = \mathfrak{gl}_l(\mathbb{C})$ and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0_{l-k} & 0 \\ 0 & X \end{pmatrix} \,\middle|\, X \in \mathfrak{gl}_k(\mathbb{C}) \right\}.$$

Let

$$\mathfrak{m} = \left\{ \begin{array}{r} {\scriptstyle l-k}\{ \\ {\scriptstyle k}\{ \end{array} \left[\begin{array}{c:c} \overbrace{U}^{l-k} & \overbrace{V}^{k} \\ \hdashline W & 0 \end{array} \right] \right\};$$

then we have

$$\mathrm{Ad}_G(h) \begin{bmatrix} U & V \\ W & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} U & V \\ W & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & h^{-1} \end{bmatrix} = \begin{bmatrix} U & Vh^{-1} \\ hW & 0 \end{bmatrix} \in \mathfrak{m}.$$

(ii) Let $G = \mathrm{SO}_l(\mathbb{C}) = \{g \in \mathrm{GL}_l(\mathbb{C}) \mid gg^T = 1, |g| = 1\}$, $H = \mathrm{SO}_k(\mathbb{C}), k \leqslant l$. Then $\mathfrak{g} = \{Z \in \mathbb{C}^{l\times l} \mid Z^T = -Z\}$. If we express Z in block matrix form as

$$Z = \begin{array}{r} {\scriptstyle l-k}\{ \\ {\scriptstyle k}\{ \end{array} \left[\begin{array}{c:c} \overbrace{U}^{l-k} & \overbrace{-Y^T}^{k} \\ \hdashline Y & V \end{array} \right],$$

where $U^T = -U$ and $V^T = -V$, then

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} \,\middle|\, V^T = -V \right\}.$$

Let

$$\mathfrak{m} = \left\{ \begin{bmatrix} U & -Y^T \\ Y & 0 \end{bmatrix} \,\middle|\, U^T = -U \right\};$$

then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and

$$\mathrm{Ad}_G(h) \begin{bmatrix} U & -Y^T \\ Y & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \begin{bmatrix} U & -Y^T \\ Y & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & h^T \end{bmatrix} = \begin{bmatrix} U & -(hY)^T \\ hY & 0 \end{bmatrix}.$$

(iii) Let

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_l = {}^{2l}\left\{ \begin{bmatrix} \overbrace{J \ \ddots \ 0}^{2l} \\ 0 \quad\quad J \end{bmatrix} \right.$$

and $G = \mathrm{Sp}_l(\mathbb{C}) = \{g \in \mathrm{GL}_{2l}(\mathbb{C}) \mid g\sigma_l g^T = \sigma_l\}$. For $k \leqslant l$ let $H = \mathrm{Sp}_k(\mathbb{C})$; then H can be identified with a subgroup of G of the form

$$\left\{ \left[\begin{array}{c|c} 1_{2(l-k)} & 0 \\ \hline 0 & h \end{array} \right] \middle| \, h\sigma_k h^T = \sigma_k \right\}.$$

Let $Z \in \mathfrak{g}$; then Z can be written in block matrix form as

$$\begin{array}{r} 2(l-k)\{ \\ 2k\{ \end{array} \left[\begin{array}{c|c} \overbrace{Z_1}^{2(l-k)} & \overbrace{-\sigma_{(l-k)} Z_2^T \sigma_k}^{2k} \\ \hline Z_2 & Z_3 \end{array} \right],$$

where $Z_1 \in \mathrm{Sp}_{l-k}(\mathbb{C})$, $Z_3 \in \mathrm{Sp}_k(\mathbb{C})$. Let $\mathfrak{h}$ denote the Lie algebra of H; then

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & Z_3 \end{bmatrix} \middle| \, Z_3\sigma_k = -\sigma_k Z_3^T \right\}.$$

Set

$$\mathfrak{m} = \left\{ \left[\begin{array}{c|c} Z_1 & -\sigma_{(l-k)} Z_2^T \sigma_k \\ \hline Z_2 & 0 \end{array} \right] \middle| \, Z_1\sigma_{(l-k)} = -\sigma_{(l-k)} Z_1^T \right\};$$

then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and

$$\begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix} \left[\begin{array}{c|c} Z_1 & -\sigma_{(l-k)} Z_2^T \sigma_k \\ \hline Z_2 & 0 \end{array} \right] \begin{bmatrix} 1 & 0 \\ 0 & h^{-1} \end{bmatrix} = \begin{bmatrix} Z_1 & -\sigma_{l-k} \left(hZ_2\right)^T \sigma_k \\ hZ_2 & 0 \end{bmatrix},$$

for all $h \in H$. Thus $H\backslash G$ is reductive.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ can be defined as the set of all left- (or right-) invariant real (or complex) analytic vector fields on G. Let $(\mathfrak{U}, \pi)$ denote the universal enveloping algebra of $\mathfrak{g}$. Then the adjoint representation of G (resp., $\mathfrak{g}$) on $\mathfrak{g}$ extends uniquely to an action of G (resp., $\mathfrak{g}$) on $\mathfrak{U}$, which we shall also denote by $\mathrm{Ad}(g)$, $g \in G$ (resp., $\mathrm{ad}(X)$, $X \in \mathfrak{g}$). Note that $\mathfrak{U}$ is identified with the algebra of all left- (or right-) invariant real (or complex) analytic differential operators on G.

Let $\mathfrak{S}(\mathfrak{g})$ denote the algebra of polynomial functions on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$; there exists a *vector- space isomorphism* $\Phi : \mathfrak{S}(\mathfrak{g}) \to \mathfrak{U}$, called the *symmetrization* defined as follows (see [5, 15.90]). If $\left(X_i\right)$ is a basis of $\mathfrak{g}$, then every polynomial $p \in \mathfrak{S}(\mathfrak{g})$ can be uniquely written as

$$p(w) = \sum_{r \leqslant s} c^{i_1 \cdots i_r} w\left(X_{i_1}\right) \cdots w\left(X_{i_r}\right) \tag{3.2}$$

where the coefficients are symmetric, i.e., $c^{i_{\sigma(1)} \cdots i_{\sigma(r)}} = c^{i_1 \cdots i_r}$, for all $\sigma \in S_r$, the symmetric group of degree r. Define

$$u = \Phi(p) = \sum_{r \leqslant s} c^{i_1 \cdots i_r} X_{i_1} \cdots X_{i_r}; \tag{3.3}$$

then it is clear that this definition is basis-independent. It follows that $\mathrm{Ad}(g)u = \sum_{r \leqslant s} c^{i_1 \cdots i_r} \mathrm{Ad}(g)X_{i_1} \cdots \mathrm{Ad}(g)X_{i_r}$. Define the coadjoint representation of G on $\mathfrak{g}^*$ by

$$(\mathrm{CoAd}(g)w)(X) = w(\mathrm{Ad}(g^{-1})X), \qquad w \in \mathfrak{g}^*, \quad X \in \mathfrak{g}, \quad g \in G.$$

The coadjoint representation extends to a representation of G on $\mathfrak{S}(\mathfrak{g})$, which we shall also denote by the same symbol. Thus

$$(\mathrm{CoAd}(g)p)(w) = p(\mathrm{CoAd}(g^{-1})w), \qquad p \in \mathfrak{S}(\mathfrak{g}), \quad w \in \mathfrak{g}^*.$$

It is straightforward to verify that $\Phi(\mathrm{CoAd}(g^{-1})p) = \mathrm{Ad}(g)\Phi(p)$, $p \in \mathfrak{S}(\mathfrak{g})$, $g \in G$. It follows that an element $u = \Phi(p)$ is invariant under the adjoint representation of G if and only if p is invariant under the coadjoint representation of G. Since G is connected, it follows that $\mathrm{Ad}(g)u = u, \forall g \in G$, if and only if $\mathrm{ad}(X)u = 0, \forall X \in \mathfrak{g}$ (see [5, 15.90]). Since $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$, and the canonical mapping $\Phi : \mathfrak{S}(\mathfrak{g}) \to \mathfrak{U}$ intertwines the representations $\mathrm{ad}_{\mathfrak{g}}$ and $\mathrm{coad}_{\mathfrak{g}}$ (see [4, prop. 2.4.9]), it follows that the subalgebra of $\mathfrak{U}$ of elements that are Ad-invariant is the *centre of* $\mathfrak{U}$. The following theorem is the main result of this paper.

Theorem 3.3. *Let $(G_k)_{k\in\mathbb{I}}$ be a chain of connected Lie groups such that $G_k \subset G_l$ and $G_k\backslash G_l$ is reductive for all $k, l \in \mathbb{I}$ with $k \leqslant l$. Let $\mathfrak{g}_k$ denote the Lie algebra of G_k, and $\mathfrak{U}_k$ the universal enveloping algebra of $\mathfrak{g}_k$. Let Ad_k denote the adjoint representation of G_k on $\mathfrak{U}_k$. Let $G^\infty = \varinjlim G_k$ denote the inductive limit of the chain $(G_k)_{k\in\mathbb{I}}$. If each $\mathfrak{U}_k$ is viewed as a vector space over* **k**, *then there exist vector-space homomorphisms $\pi^l_k : \mathfrak{U}_l \to \mathfrak{U}_k, l \geqslant k$, such that the family $\{\mathfrak{U}_k; \pi^l_k\}$ is an inverse spectrum over the index set $\mathbb{I}$ with connecting morphisms π^l_k. If $\mathfrak{U}_\infty = \varprojlim \mathfrak{U}_k$ is the inverse (or projective) limit of the inverse spectrum $\{\mathfrak{U}_k; \pi^l_k\}$, then $\mathfrak{U}_\infty$ is a vector space over* **k**. *There is a well-defined representation of G^∞ on $\mathfrak{U}_\infty$ which we shall call the adjoint representation and we shall denote by Ad_{G^∞}. If for each $k \in \mathbb{I}$ we let $\mathfrak{C}_k$ denote the vector subspace of Ad_k-invariants in $\mathfrak{U}_k$, and $\mathfrak{C}$ denote the subspace of Ad_{G^∞}-invariants in $\mathfrak{U}_\infty$, then $\mathfrak{C} = \varprojlim \mathfrak{C}_k$.*

Proof. We can choose a complementary subspace $\mathfrak{m}^l_k$ of $\mathfrak{g}_k$ in $\mathfrak{g}_l$ such that $\mathrm{Ad}_{G_l}(h)\mathfrak{m}^l_k \subset \mathfrak{m}^l_k$ $(h \in G_k)$. Let n (resp., p) be the dimension of $\mathfrak{g}_l$ (resp., $\mathfrak{g}_k$), and $(X_1, \ldots, X_n)$ be a basis of $\mathfrak{g}_l$ such that $(X_1, \ldots, X_p)$ is a basis of $\mathfrak{g}_k$ and $(X_{p+1}, \ldots, X_n)$ is a basis of $\mathfrak{m}^l_k$. Then $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, form a basis for the vector space $\mathfrak{U}_l$. Hence every element $u \in \mathfrak{U}_l$ can be uniquely written in the form

$$u = \sum_{\alpha_1,\ldots,\alpha_p\in\mathbb{N}} u_{\alpha_1\cdots\alpha_p} X_{p+1}^{\alpha_{p+1}} \cdots X_n^{\alpha_n}, \qquad \text{where} \quad u_{\alpha_1\cdots\alpha_p} \in \mathfrak{U}_k. \tag{3.4}$$

It follows that every element $u \in \mathfrak{U}_l$ can be uniquely written in the form

$$u = u_k + v_k, \tag{3.5}$$

where $u_k \in \mathfrak{U}_k$, and v_k is a sum of the form (3.4) such that at least one of the α_{p+j}, $1 \leqslant j \leqslant n - p$, is not zero. Let $\mathfrak{V}^l_k$ be the complementary subspace of the subspace $\mathfrak{U}_k$ in the vector space $\mathfrak{U}_l$ spanned by the elements of the form v_k. For $h \in G_k$ we have

$$\mathrm{Ad}_{G_l}(h)v_k = \sum_{\alpha_1,\ldots,\alpha_p\in\mathbb{N}} \big(\mathrm{Ad}_{G_l}(h)u_{\alpha_1\cdots\alpha_p}\big)\big(\mathrm{Ad}_{G_l}(h)X_{p+1}^{\alpha_{p+1}}\big) \cdots \big(\mathrm{Ad}_{G_l}(h)X_n^{\alpha_n}\big) \tag{3.6}$$

The condition $\mathrm{Ad}_{G_l}(h)\mathfrak{m}^l_k \subset \mathfrak{m}^l_k$ and equation (3.6) imply that $\mathrm{Ad}_{G_l}(h)\mathfrak{V}^l_k \subset \mathfrak{V}^l_k$.

Let $\pi^l_k : \mathfrak{U}_l \to \mathfrak{U}_k$ be defined by

$$\pi^l_k(u) = u_k \qquad (u \in \mathfrak{U}_l). \tag{3.7}$$

Then obviously π^l_k is a vector-space homomorphism. Now consider $\mathfrak{U}_j, \mathfrak{U}_k$ and $\mathfrak{U}_l$ with $j \leqslant k \leqslant l$. Let $u \in \mathfrak{U}_l$; then equation (3.5) implies that

$$u = u^l_k + v^l_k, \qquad \text{where} \quad u^l_k \in \mathfrak{U}_k \qquad \text{and} \qquad v^l_k \in \mathfrak{V}^l_k.$$

Again by equation (3.5), u_k^l can be uniquely written as a direct sum $u_k^l = \left(u_k^l\right)_j + \left(v_k^l\right)_j$, where $\left(u_k^l\right)_j \in \mathfrak{U}_j$ and $\left(v_k^l\right)_j \in \mathfrak{V}_j^k$. It follows that

$$u = \left(u_k^l\right)_j + \left(\left(v_k^l\right)_j + v_k^l\right), \qquad \text{where} \quad \left(v_k^l\right)_j + v_k^l \in \mathfrak{V}_j^l.$$

This implies that $\pi_j^l(u) = \left(u_k^l\right)_j = \pi_j^k \circ \pi_k^l(u)$ for every $u \in \mathfrak{U}_l$ and $j \leqslant k \leqslant l$. Thus the family $\left\{\mathfrak{U}_k; \pi_k^l\right\}$ is an inverse spectrum over the index set $\mathbb{I}$, and its inverse limit $\mathfrak{U}_\infty$ is a vector space over $\mathbf{k}$, and the canonical maps $\pi_k : \mathfrak{U}_\infty \to \mathfrak{U}_k$ are vector-space homomorphisms (see [15, p 404]). From equations (3.5)–(3.7) it follows that

$$\mathrm{Ad}_{G_k}(g)\big(\pi_k^l(u)\big) = \pi_k^l\big(\mathrm{Ad}_{G_l}(g)u\big), \qquad \forall\, g \in G_k, \quad u \in \mathfrak{U}_l. \tag{3.8}$$

Equation (3.8) implies that there is a well-defined action of G^∞ on $\mathfrak{U}_\infty$ (see [15, lemma 3.1, p 410]) which we shall call the adjoint representation of G^∞ on $\mathfrak{U}_\infty$. From [15, theorem 3.4, p 411] it follows that the subspace $\mathfrak{C}$ of the vector space $\mathfrak{U}_\infty$ of all Ad_{G^∞}-invariants is the inverse limit of the vector subspaces $\mathfrak{C}_k$ of all Ad_k-invariants in $\mathfrak{U}_k$. □

Corollary 3.4. *There exists a well-defined representation of the Lie algebra $\mathfrak{g}^\infty$ on the vector space $\mathfrak{U}_\infty$. This representation is called the 'adjoint representation' and is denoted by* $\mathrm{ad}_{\mathfrak{g}^\infty}$. *Moreover an element $u \in \mathfrak{U}_\infty$ is invariant under* Ad_{G^∞} *if and only if it is annihilated by* $\mathrm{ad}_{\mathfrak{g}^\infty}(X)$ *for all $X \in \mathfrak{g}^\infty$.*

Proof. In definition 3.1 if the closed subgroup H of G is also connected then it follows from [5, 5.90] that equation (3.1) is equivalent to the following equation:

$$\mathrm{ad}_G(Y)\mathfrak{m} \subset \mathfrak{m} \qquad (\forall Y \in \mathfrak{h}).$$

This equation, in turn, implies that equation (3.8) is equivalent to the following equation:

$$\mathrm{ad}_{\mathfrak{g}_k}(Y)\big(\pi_k^l(u_l)\big) = \pi_k^l\big(\mathrm{ad}_{\mathfrak{g}_l}(Y)u_l\big) \qquad (\forall Y \in \mathfrak{g}_k, u_l \in \mathfrak{U}_l, l \geqslant k). \tag{3.9}$$

Then an argument similar to the one used in lemma 3.1 of [15] shows that $\mathrm{ad}_{\mathfrak{g}^\infty}$ is well defined. It is straightforward to verify that an element $u = (u_k) \in \mathfrak{U}_\infty$ is annihilated by $\mathrm{ad}_{\mathfrak{g}^\infty}(X)$ for all $X \in \mathfrak{g}^\infty$ if and only if u_k is annihilated by $\mathrm{ad}_{\mathfrak{g}_k}(X_k)$, for all $X_k \in \mathfrak{g}_k$. On the other hand, we have shown that if G_k is connected then $\mathrm{Ad}_{G_k}(g_k)u_k = u_k$ if and only if $\mathrm{ad}_{\mathfrak{g}_k}(X_k)u_k = 0$ for all $g_k \in G_k$, and $X_k \in \mathfrak{g}_k$. But lemma 3.2 of [15] states that $\mathrm{Ad}_{G^\infty}(g)u = u$ if and only if $\mathrm{Ad}_{G_k}(g_k)u_k = u_k$ for all $g \in G^\infty$, and $g_k \in G_k$. From this the second statement of the corollary follows immediately. □

Remark 3.5. Since $\mathrm{ad}_{\mathfrak{g}_k}(X_k)u_k = X_k u_k - u_k X_k$ (see, e.g., [4, prop. 2.4.9]), an element $u_k \in \mathfrak{U}_k$ is annihilated by $\mathrm{ad}_{\mathfrak{g}_k}(X_k)$ $(\forall X_k \in \mathfrak{g}_k)$ if and only if it commutes with all $X_k \in \mathfrak{g}_k$. It follows from corollary 3.4 that an element $u \in \mathfrak{U}_\infty$ is Ad_{G^∞}-invariant if and only if it commutes with all $X \in \mathfrak{g}^\infty$. In physics, if G is a symmetry group of some physical system, then the spectra of the G-invariant operators determine the observable quantum numbers of the system. For this reason, we call the subspace $\mathfrak{C}$ of Ad_{G^∞}-invariants in $\mathfrak{U}_\infty$ the space of *generalized Casimir invariants*.

We preserve the hypotheses and notation of theorem 3.3. Let $\mathfrak{S}_k$ denote the algebra of polynomial functions on $\mathfrak{g}_k^*$, and $\Phi_k : \mathfrak{S}_k \to \mathfrak{U}_k$ denote the symmetrization isomorphism of vector spaces. Let CoAd_{G_k} denote the coadjoint representation of G_k on $\mathfrak{S}_k$ and J_k the subspace of $\mathfrak{S}_k$ of all CoAd_{G_k}-invariants. Then the family $\left\{\mathfrak{S}_k; \mu_k^l\right\}$ is an inverse spectrum, where $\mu_k^l : \mathfrak{S}_l \to \mathfrak{S}_k, l \geqslant k$, is the connecting homomorphism defined by equation (2.4) of [15]. Let $\mathfrak{S}_\infty = \varprojlim \mathfrak{S}_k$; then by lemma 3.1 of [15] there is a well-defined action, denoted by CoAd_{G^∞}, of G^∞ on $\mathfrak{S}_\infty$. Let J denote the subspace of CoAd_{G^∞}-invariants in $\mathfrak{S}_\infty$ and $J_\infty = \varprojlim J_k$. Then according to theorem 3.4 of [15] $J_\infty = J$, and hence J is closed in A_∞.

Theorem 3.6. *There exists a well-defined vector-space isomorphism $\Phi_\infty : \mathfrak{S}_\infty \to \mathfrak{U}_\infty$ which intertwines the representations CoAd_{G^∞} and Ad_{G^∞}. Moreover, Φ_∞ maps J isomorphically onto $\mathfrak{C}$.*

Proof. For the existence and uniqueness of such a map Φ_∞ see [2, section 12]. It also follows from [3, p 131] that Φ_∞ is an intertwining monomorphism. It remains to show that Φ_∞ is surjective since the inverse limit of an inverse system of surjective maps is not necessarily surjective (see [3, p 132]). We have the following commutative diagram of vector-space homomorphisms:

$$\begin{array}{ccc} \mathfrak{S}_\infty & \xrightarrow{\Phi_\infty} & \mathfrak{U}_\infty \\ \mu_l \downarrow & & \downarrow \pi_l \\ \mathfrak{S}_l & \xrightarrow{\Phi_l} & \mathfrak{U}_l \\ \mu_k^l \downarrow & & \downarrow \pi_k^l \\ \mathfrak{S}_k & \xrightarrow{\Phi_k} & \mathfrak{U}_k \end{array}$$

where μ_l and π_l are canonical homomorphisms, and $k \leqslant l$. Let $u \in \mathfrak{U}_\infty$; then $u = (u_k)$, where $u_k = \pi_k(u), \forall k \in \mathbb{I}$. Since Φ_k is an isomorphism there exists a unique $p_k \in \mathfrak{S}_k$ such that $\Phi_k(p_k) = u_k$. Set $p = (p_k)$ and let us show that p is a thread; i.e., $\mu_k^l(p_k) = p_k$ whenever $l \geqslant k$. We have

$$\pi_k^l \circ \Phi_l(p_l) = \Phi_k \circ \mu_k^l(p_l).$$

By definition $\pi_k^l(\Phi_l(p_l)) = \pi_k^l(u_l) = u_k$. So

$$\Phi_k\big(\mu_k^l(p_l)\big) = u_k.$$

Since Φ_k is an isomorphism we have $\mu_k^l(p_l) = p_k$. Thus $(p_k) = p \in \mathfrak{S}_\infty$. Let $u' = \Phi_\infty(p)$; then from the diagram above we have

$$\pi_l \circ \Phi_\infty(p) = \Phi_l \circ \mu_l(p) = \Phi_l(p_l) = u_l.$$

But $\pi_l \circ \Phi_\infty(p) = \pi_l(u') = u_l'$. Thus $u_l' = u_l, \forall l \in \mathbb{I}$; therefore $u' = u$, and Φ_∞ is surjective. It is clear that Φ_∞ maps J isomorphically onto $\mathfrak{C}$. □

Definition 3.7. *A thread $u = (u_j) \in \mathfrak{U}_\infty$ is called a stationary thread at level k if there exists a least index $k \in \mathbb{I}$ such that $u_l = u_k$ for all $l \geqslant k$. Note that an element $u \in \mathfrak{U}^\infty$ can be considered as a stationary thread of $\mathfrak{U}_\infty$. Indeed, let k be the least index such that $u \in \mathfrak{U}_k$; then since $\mathfrak{U}_k \subset \mathfrak{U}_l$ for all $l \geqslant k$ it follows that the thread (u_j) with $u_j = u$ for all $j \geqslant k$ is a stationary thread at the level k. For the terminology and notation of the following theorem see [15].*

Theorem 3.8. *For each $k \in \mathbb{I}$ let $\mathfrak{C}_k$ denote the vector space of Ad_k-invariants in $\mathfrak{U}_k$. If for each $k \in \mathbb{I}$ there exists an algebraic basis $\{u_k^\alpha\}_{\alpha\in\Lambda_k}$ of $\mathfrak{C}_k$ which satisfies the following conditions:*

(i) the index sets Λ_k form an increasing chain $\Lambda_k \subset \Lambda_l$ for $k \leqslant l$, and
(ii) each thread $u^\alpha = (u_k^\alpha) = \varprojlim u_k^\alpha$, $\alpha \in \Lambda = \bigcup_{k\in\mathbb{I}} \Lambda_k$, is non-stationary,

then the centre of $\mathfrak{U}^\infty$ is trivial.

Proof. By theorem 3.10 of [15] the set $\{u^\alpha\}_{\alpha\in\Lambda}$ is an inverse limit basis for $\mathfrak{C}$. If u is an element of the centre of $\mathfrak{U}^\infty$ then $u \in \mathfrak{U}_k$ for some $k \in \mathbb{I}$. Therefore u must be a stationary thread at the level k, i.e., $u_l = u_k \equiv u$ for $k \leqslant l$. Since u is at the centre of $\mathfrak{U}^\infty$, u belongs to $\mathfrak{C}_l$ for all $l \geqslant k$. Thus for each $l \geqslant k$, $u = p_l\left(\left\{u_l^\alpha\right\}_{\alpha\in\Lambda_l}\right)$ where p_l is a non-commutative polynomial in $\left\{u_l^\alpha\right\}_{\alpha\in\Lambda_l}$. Conditions (i) and (ii) of the theorem imply that this is possible if and only if u is a scalar. □

Remark 3.9. If $(G_k)_{k\in\mathbb{I}}$ is a chain of connected and semi-simple Lie groups such that $G_k \subset G_l$ and $G_k\backslash G_l$ is reductive for all $k, l \in \mathbb{I}$ with $k \leqslant l$ then theorem 7.3.8 of [4] (see also theorem 4.9.4 of [16], or theorem 2, p 253 of [1]) implies that conditions (i) and (ii) of theorem 3.8 are met. Thus the centre of $\mathfrak{U}^\infty$ is trivial. In particular, we have the following.

Corollary 3.10. *Let* $(\mathfrak{g}_k)_{k\in\mathbb{I}}$ *be a chain of classical simple Lie algebras over* $\mathbb{C}$. *Then the centre of* $\mathfrak{U}^\infty$ *is trivial.*

4. Conclusion

We have developed a general theory of Casimir invariants for a class of inductive limits of Lie groups. In [13] we gave an explicit inverse-limit basis for the Casimir invariants of the group $\mathrm{GL}^\infty(\mathbb{C})$. These Casimir operators act on a generalized Bargman–Segal–Fock space. This led to a generalized Racah–Chevalley theorem (see theorem 2, p 253 of [1]) for $\mathrm{GL}^\infty(\mathbb{C})$. This was used, in turn, in [9] to decompose tensor products of tame representations of $\mathrm{U}(\infty)$. It is our intention to apply this theory to tame representations of all infinite-dimensional classical groups.

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